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CHARACTERIZATION OF L^p-SOLUTIONS FOR THE TWO-SCALE DILATION EQUATIONS*

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Abstract. We give a characterization of the existence of compactly supported L^p-solutions, $1 \leq p < \infty$, for the two-scale dilation equations. For the L^2-case, the condition reduces to the determination of the spectral radius of a certain matrix in terms of the coefficients, which can be calculated through a finite step algorithm. For the other cases, we implement the characterization by the four-coefficient dilation equation and obtain some simple sufficient conditions for the existence of the solutions. The results are compared with known ones.

Key words. cascade algorithm, compactly supported L^p-solutions, dilation equation, Fourier transformation, iteration, spectral radius, wavelet

AMS subject classifications. 26A15, 26A18, 39A10, 42A05

1. Introduction.

A *two-scale dilation equation* is a functional equation of the form

$$f(x) = \sum_{n=0}^{N} c_n f(\alpha x - \beta_n), \tag{1.1}$$

where f: $\mathbf{R} \longrightarrow \mathbf{R}$ (or $\mathbf{C}$), $\alpha > 1$, $\beta_0 < \beta_1 < \cdots < \beta_N$ are real constants, and c_n are real (or complex) constants. The equation is called a *lattice* two-scale dilation equation if

$$f(x) = \sum_{n=0}^{N} c_n f(kx - n) \tag{1.2}$$

for an integer $k \geq 2$. A special case of the functional equation ($k = 3$, $N = 4$, and $c_n = 1,\ 2/3,\ 1/3,\ 1/3,\ 1$) was first studied by de Rham [dR] as an example of a continuous nowhere differentiable function. Recently this equation has attracted a lot of attention, especially for the lattice case with $k = 2$. In wavelet theory, the study of multiresolution and the search of various orthogonal, compactly supported wavelets has lead to the investigation of the existence, uniqueness, and smoothness of such continuous integrable solutions (see the work of Cohen, Colella, Daubechies, Heil, Lagarias, Lawton, Mallat and Meyer; see the survey paper [H]). The equation also plays an important role in the "subdivision schemes" and "interpolation schemes" of constructing continuous spline curves, surfaces, and fractal objects (see the work of Cavaretta, Dahmen, Deslauriers, Dubuc, Dye, Gregory, Levin, Michelli, Prautzsch; see [DL1] and [DL2] for an historical development and references).

The general two-scale (in fact multiscale) dilation equation (1.1) arises in the consideration of self-similar measures (Hutchinson [Hu]), and the singularity of the measures induced by the infinite Bernoulli convolutions. The latter has been studied

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for a long time and the question is still unsettled (see the work of Erdös, Garcia, Jessen, Salem, Wintner; see [L1] and [L2] for some recent developments and remarks). In another direction, Strichartz [JRS], [Str] studied the asymptotic behavior of the Fourier transformation of such distributions, and made many interesting observations on the averages with respect to some fractional powers.

There are two major approaches to the equation: the Fourier method (the frequency domain approach) and the iteration method (the time-domain approach). The Fourier transformation converts the functional equation to the form

$$\hat{f}(\xi) = A \prod_{j=0}^{\infty} p(\alpha^{-j}\xi),$$

where $p(\xi) = \frac{1}{\alpha}\sum_{n=0}^{N} c_n e^{i\beta_n \xi}$. Using this, Daubechies and Lagarias [DL1] proved that for $\Delta = \alpha^{-1}\sum c_n$,

(i) if $|\Delta| < 1$ or $\Delta = -1$, then (1.1) has no integrable solution;

(ii) if $\Delta = 1$ then it has *at most* one nonzero integrable solution;

(iii) if $|\Delta| > 1$ and if an integrable solution f exists, then $\Delta = \alpha^m$ for some nonnegative integer m. The dilation equation obtained by replacing the coefficients $\{c_n\}$ with $\{\alpha^{-m}c_n\}$ has a nonzero integrable solution g, and for suitable choice of normalization,

$$\frac{d^m}{dx^m} g(x) = f(x) \qquad \text{a.e.}$$

For an integrable solution, the above result essentially reduces the coefficients of the equation to the special case

$$\sum c_n = \alpha.$$

By using the Fourier transform of f and the Paley–Wiener theorem, it was also proved in [DL1] that f has compact support in $[0, \beta_N N/(\alpha - 1)]$. The Fourier method, however, does not give sharp criteria for the existence of L^1-solutions in terms of the coefficients $\{c_n\}$. Some partial results are given in [La] and [M].

The iteration method is restricted to the lattice case. It applies particularly well in the case of compactly supported solutions. The basic idea is to identify a given function f supported by $[0, N]$ with the vector-valued function

$$\mathbf{f}(x) = [f(x), f(x+1), \ldots, f(x+(N-1))]^t, \quad x \in [0,1],$$

and to use the right side of the dilation equation to construct two $N \times N$ matrices T_0 and T_1 (see §2 for details). A constant vector v is used as the initial condition, followed by iteration with the matrices T_0 and T_1 (the *cascade algorithm*). The limit, if the sequence converges, will be the solution of the dilation equation. Such an approach was used by Daubechies and Lagarias [DL2], and independently by Michelli and Prautzsch [MP]. It was also used by Berger and Wang [BW1], and Collela and Heil [CH1] and [CH2].

For two given matrices A_0 and A_1, Rota and Strang [RS] and Strang [S] defined the *joint spectral radius* of A_0, A_1 by

$$\hat{\rho}(A_0, A_1) = \limsup_{m \to \infty} \lambda_m(A_0, A_1),$$

where

$$\lambda_m(A_0, A_1) = \max_{|J|=m} ||A_J||^{\frac{1}{m}}$$

with $J = (j_1, \ldots, j_l)$, $A_J = A_{j_1} \cdots A_{j_l}$, $j_i = 0$ or 1. A useful sufficient condition for the existence of solutions is given in [DL2] and [BW1].

THEOREM 1.1. *For $k = 2$, $\sum c_{2n} = \sum c_{2n+1} = 1$, let*

$$H = \left\{ u = [u_1, \ldots, u_N]^t : \quad \sum u_i = 0 \right\}.$$

If $\hat{\rho}(T_0|_H, T_1|_H) < 1$, then the equation has a nonzero continuous integrable solution.

Colella and Heil [CH1] and [CH2] also showed that the condition is "essentially" necessary. More recently Wang [W] introduced the notion of *mean spectral radius*:

$$\bar{\rho}(T_0, T_1) = \limsup_{m \to \infty} \frac{1}{2} \left(\sum_{|J|=m} ||T_J|| \right)^{\frac{1}{m}}.$$

He proved, among other interesting results, that if $\sum c_{2n} = \sum c_{2n+1} = 1$ and $\bar{\rho}(A_0, A_1) < 1$, where

$$T_i \approx \begin{bmatrix} 1 & 0 \\ b_i & A_i \end{bmatrix}, \qquad i = 0, 1,$$

then a nonzero integrable solution exists.

This characterization in terms of the joint spectral radius, although elegant, is difficult to evaluate in practice. By using a geometric convergence consideration and a different iteration argument, Pan [P] gives a simple sufficient condition for the existence of compactly supported L^p-solutions of the functional equation (1.2) with four coefficients.

In this paper we will continue to study the existence of the compactly supported L^p-solutions of

$$f(x) = \sum_{n=0}^{N} c_n f(2x - n), \tag{1.3}$$

using the cascade iteration algorithm with the matrices T_0 and T_1. The regularity of such solutions will be dealt with in a forthcoming paper. Note that in the previous literature, one always starts with an initial condition that is, in a certain sense, quite arbitrary (for example, a spline function or $\chi_{[0,1]}$). Our fundamental observation is the following proposition.

PROPOSITION 1.2. *Suppose $1 \le p < \infty$ and $\sum c_n = 2$. Let f be a compactly supported L^p-solution of* (1.3) *and let*

$$v = \left[\int_0^1 f, \ldots, \int_{N-1}^N f \right]^t.$$

Then v is an eigenvector of $(T_0 + T_1)$ corresponding to the eigenvalue 2.

It follows that we can start with the iteration algorithm on the 2-eigenvector of $T_0 + T_1$, and the convergence condition will be imposed *only* on the subspace involved with such eigenvector. This allows us to obtain sharper results. The basic theorem is as follows.

THEOREM 1.3. *Suppose $1 \le p < \infty$. Then equation* (1.3) *has a nonzero compactly supported L^p-solution if and only if there exists a* 2*-eigenvector v of $(T_0 + T_1)$ such that*

$$\frac{1}{2^l} \sum_{|J|=l} ||T_J(T_0 - I)v||^p \longrightarrow 0 \quad \text{as } l \longrightarrow \infty.$$

For computational purposes we let $H(\tilde{v})$ be the subspace in $\mathbf{R}^n$ generated by the $T_J\tilde{v}$'s for all J, where $\tilde{v} = (T_0 - I)v$, and let $\{v_1, \ldots, v_k\}$ be a basis; then the above condition is equivalent to the existence of an integer l such that

$$\frac{1}{2^l} \sum_{|J|=l} ||T_J v_i||^p < 1 \tag{1.4}$$

for all v_i, $i = 1, \ldots, k$.

The above results, as well as some corollaries and remarks, are proved in §2. A slight improvement of Theorem 1.3 under the condition that the coefficients satisfy the "m-sum rules" (see (2.7)) is also considered.

In §3, we consider the equation for the three-coefficient ($N = 2$) and the four-coefficient ($N = 3$) cases. For the first case we obtain a complete characterization of the compactly supported L^p-solutions. The second case is less trivial; it contains the well-known Daubechies wavelet D_4 [D], and has been studied in detail in [H] and [P]. By using the basic theorem, we are able to derive some simple criteria for such solutions to exist.

In §4 we give an improvement of Theorem 1.3 for the L^2-case. In this case, the left-hand side of (1.4) can be calculated and leads to an explicit expression of an $N \times N$ matrix W (Lemma 4.1, Proposition 4.3). Under a stronger assumption on the coefficients

$$\sum c_{2n} = \sum c_{2n+1} = 1, \tag{1.5}$$

we show that the matrix W has an eigenvalue 2; (1.4), and hence the existence of the compactly supported L^2-solution, is essentially equivalent to the fact that all other eigenvalues of W are less than 2 (Theorem 4.4 and Proposition 4.6). For the four-coefficient case we obtain a complete characterization of the existence of the compactly supported L^2-solutions (Theorem 4.8).

There are different criteria for the existence of L^2-solutions; e.g., see [E], [Her1], [Her2], and [V]. Their approach is via a Fourier method which is quite different from ours (see Remark 9 in §4).

In [CH1], Collela and Heil used (c_0, c_3) as free parameters for the four-coefficient case satisfying $c_0 + c_2 = c_1 + c_3 = 1$, and plotted different domains in $\mathbf{R}^2$ that admit or do not admit solutions. We conclude our study with an appendix for displaying our result and some other well-known results with the same kind of plots.

2. The basic theorems. Throughout this paper we will consider the compactly supported L^p-solutions, $1 \le p < \infty$, of the functional equation

$$f(x) = \sum_{n=0}^{N} c_n f(2x - n). \tag{2.1}$$

The general lattice case can be handled similarly (see (2.5)). For convenience we let $c_n = 0$ if $n \notin \{0, \ldots, N\}$. Our basic assumption on the coefficients is $\sum c_n = 2$. For some cases we will also assume that $\sum c_{2n} = \sum c_{2n+1} = 1$. We will further restrict the c_n's and the function f to be real valued, though there is no difficulty in extending our method to the complex case.

It is known that if an L^1-solution exists, then it is necessarily unique, and is supported by $[0, N]$ [DL1]. This is not true if $1 < p < \infty$, since the Hilbert transformation of such solution is again an L^p-solution [H]. We will use *L^p_c-solution* to denote the compactly supported L^p-solution. An L^p_c-solution must be integrable, and hence supported by $[0, N]$.

Formally, the solution f is obtained by taking the limit of $\mathbf{S}^n(g)$ for a suitable function g on $\mathbf{R}$, where

$$\mathbf{S}(g)(x) = \sum_{n=0}^{N} c_n g(2x - n).$$

In some previous papers [BW], [CH1], [CH2], [DL2], [MP], [W], it is found that the analysis is a lot more convenient if we convert the involved functions into vector forms and the operator $\mathbf{S}$ into a matrix operator. For this we let

$$T_0 = [c_{2i-j-1}]_{1\leq i,j\leq N} = \begin{bmatrix} c_0 & 0 & 0 & \dots & 0 \\ c_2 & c_1 & c_0 & \dots & 0 \\ c_4 & c_3 & c_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \dots & c_{N-1} \end{bmatrix},$$

$$T_1 = [c_{2i-j}]_{1\leq i,j\leq N} = \begin{bmatrix} c_1 & c_0 & 0 & \dots & 0 \\ c_3 & c_2 & c_1 & \dots & 0 \\ c_5 & c_4 & c_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \dots & c_N \end{bmatrix}.$$

For any g defined on $\mathbf{R}$ vanishing outside $[0, N]$, we decompose g into N pieces and form a vector function as follows: let $g_i(x) = g(x+i)\chi_{[0,1)}, i = 0, 1, \dots N-1$, and define a vector function $\mathbf{\Phi}(g) = \mathbf{g} : \mathbf{R} \longrightarrow \mathbf{R}^N$ by

$$\mathbf{g}(x) = \begin{cases} [g_0(x), g_1(x), \dots, g_{N-1}(x)]^t & \text{if } x \in [0,1), \\ 0 & \text{if } x \notin [0,1). \end{cases}$$

Here we use v^t to denote the transpose of a vector v. Let $||\cdot||$ be any fixed norm on $\mathbf{R}^N$ and define, as usual, $||\mathbf{g}|| = ||\mathbf{g}||_{L^p} = (\int_0^1 ||\mathbf{g}(x)||^p dx)^{1/p}$, so $g \in L^p[0, N]$ if and only if $\mathbf{g} \in L^p([0,1], \mathbf{R}^N)$. Note that if we take the l^p-norm on $\mathbf{R}^N$, then $||\mathbf{g}||_{L^p} = ||g||_{L^p}$.

Let $\mathbf{T}$ be an operator defined on the vector-valued functions $\mathbf{g}$ by

$$(\mathbf{Tg})(x) = T_0 \cdot \mathbf{g}(\phi_0^{-1}(x)) + T_1 \cdot \mathbf{g}(\phi_1^{-1}(x)),$$

where $\phi_0(x) = \frac{1}{2}x$, $\phi_1(x) = \frac{1}{2}x + \frac{1}{2}$. Equivalently, $\mathbf{T}$ is given by

$$(\mathbf{Tg})(x) = \begin{cases} T_0 \cdot \mathbf{g}(2x) & \text{if } x \in [0, \frac{1}{2}), \\ T_1 \cdot \mathbf{g}(2x-1) & \text{if } x \in [\frac{1}{2}, 1), \end{cases}$$

and $(\mathbf{Tg})(x) = 0$ if $x \notin [0,1)$. If we iterate the operator $\mathbf{T}$ on $\mathbf{g}$ repeatedly and obtain a formal limit $\mathbf{f}$, then $\mathbf{f}$ will satisfy

$$\mathbf{f}(x) = T_0 \cdot \mathbf{f}(\phi_0^{-1}(x)) + T_1 \cdot \mathbf{f}(\phi_1^{-1}(x)). \tag{2.2}$$

PROPOSITION 2.1. *Let f be supported by $[0, N]$, and let $\mathbf{\Phi}(f) = \mathbf{f}$ be defined as above; then*

$$\mathbf{\Phi S}(f) = \mathbf{T\Phi}(f).$$

Moreover, f is an L^p_c-solution of (2.1) if and only if $\mathbf{f} \in L^p([0,1], \mathbf{R}^N)$ *and* $\mathbf{f} = \mathbf{T}\mathbf{f}$, *i.e.,* $\mathbf{f}$ *satisfies equation (2.2).*

Proof. The proof of the commutativity of the operators only involves a direct computation of x in $[i, i+\frac{1}{2}]$ and $[i+\frac{1}{2}, i+1]$. The second part is a consequence of the first part, making use of the fact that if the solution has compact support, then it must be contained in $[0, N]$. □

We begin with some simple considerations of the eigenproperties of T_0 and T_1. Unless otherwise specified, eigenvector will mean *right* eigenvector. By a λ-eigenvector of a matrix M, we will mean an eigenvector of M corresponding to the eigenvalue λ. The following proposition is known.

PROPOSITION 2.2. *If* $\sum c_n = 2$, *then 2 is an eigenvalue of* $(T_0 + T_1)$ *with left eigenvector* $[1, \ldots, 1]$.

Furthermore, if $\sum c_{2n} = \sum c_{2n+1} = 1$, then 1 is an eigenvalue of T_0 and T_1 with $[1, \ldots, 1]$ as a left eigenvector.

Proof. We need only observe that in the matrix $(T_0 + T_1)$, each column has sum equal to 2. The proof of the second statement is similar. □

It follows that the *right* 2-eigenvector of (T_0+T_1) exists also; it will play a central role in the existence of the solution of (2.1). Let f_Δ be the average of f over an interval Δ, i.e.,

$$f_\Delta = \frac{1}{|\Delta|} \int_\Delta |f|,$$

where $|\Delta|$ is the length of Δ.

PROPOSITION 2.3. *Let f be an L^p_c-solution of (2.1); let* $v = [f_{[0,1]}, \ldots, f_{[N-1,N]}]^t$ *be the vector defined by the average of f on the N subintervals as indicated. Then v is a 2-eigenvector of* $(T_0 + T_1)$.

Proof. Since $\mathbf{f} = \mathbf{T}\mathbf{f}$, i.e.,

$$\mathbf{f}(x) = \begin{cases} T_0 \cdot \mathbf{f}(2x) & \text{if } x \in [0, \frac{1}{2}), \\ T_1 \cdot \mathbf{f}(2x-1) & \text{if } x \in [\frac{1}{2}, 1), \end{cases}$$

when we integrate the expression over $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ separately, we have

$$\begin{bmatrix} f_{[0,\frac{1}{2}]} \\ \vdots \\ f_{[N-1,N-\frac{1}{2}]} \end{bmatrix} = T_0 v, \qquad \begin{bmatrix} f_{[\frac{1}{2},1]} \\ \vdots \\ f_{[N-\frac{1}{2},N]} \end{bmatrix} = T_1 v.$$

On the other hand, note that on each interval $[i, i+1]$, the average satisfies

$$f_{[i,i+\frac{1}{2}]} + f_{[i+\frac{1}{2},i+1]} = 2 f_{[i,i+1]};$$

hence we conclude that $(T_0 + T_1)v = 2v$. □

We will show, under suitable conditions, that the 2-eigenvector of $(T_0 + T_1)$ actually defines a step function that generates the solution of (2.1). This is done by iterating with the operator $\mathbf{T}$, and is itself the average vector of the solution. For this purpose, we need to introduce some notation for the indices: For any $k \geq 1$, let

$$J = (j_1, \ldots, j_k), \qquad \text{where } j_i = 0 \text{ or } 1, \quad i = 1, 2, \ldots, k,$$

and set $J = \emptyset$ if $k = 0$ for convenience; we will use $|J|$ to denote the length of J, and let

$$\Lambda = \{J : \quad |J| = k, \quad k = 0, 1, 2, \ldots\}$$

denote the class of indices. For $J, J' \in \Lambda$, we let $(J, J') = (j_1, \ldots, j_k, j'_1, \ldots, j'_{k'})$. Let I be the interval $[0, 1)$; I_J will denote the dyadic interval $\phi_{j_1} \circ \phi_{j_2} \cdots \circ \phi_{j_k}([0, 1))$. For example, $I_0 = [0, \frac{1}{2})$, $I_1 = [\frac{1}{2}, 1)$, and $I_J = I_{(j_1, \ldots, j_k)} = [a, b)$, where

$$a = \frac{j_1}{2} + \frac{j_2}{2^2} + \cdots + \frac{j_k}{2^k}, \qquad b = a + \frac{1}{2^k}.$$

It follows that $I_{(J,0)} \cup I_{(J,1)} = I_J$ and $I_{(J,J')} \subseteq I_J$ for any $J, J' \in \Lambda$. The matrix T_J represents the product $T_{j_1} \ldots T_{j_k}$ and $T_\emptyset$ is the identity matrix.

LEMMA 2.4. *Let* $\mathbf{f}_0(x) = v$ *for* $x \in [0, 1)$, *and* $\mathbf{f}_{k+1} = \mathbf{T}\mathbf{f}_k$, $k = 0, 1, \ldots$; *then* $\mathbf{f}_k(x) = T_J v$ *for each* $x \in I_J$.

Moreover, if f *is an* L^p_c*-solution of* (2.1) *and* v *is the average vector of* f *defined in Proposition* 2.3, *then*

$$\mathbf{f}_k(x) = T_J v = [f_{I_J}, f_{(I_J+1)}, \ldots, f_{(I_J+N-1)}]^t,$$

where $(I_J + j)$ *is the interval* $\{x + j : x \in I_J\}$. *Also,* $\mathbf{f}_k \longrightarrow \mathbf{f} = \mathbf{\Phi}(f)$ *in* $L^p([0, 1], \mathbf{R}^N)$.

Proof. We will use induction to show that $\mathbf{f}_k(x) = T_J v$ for $x \in I_J$ with $|J| = k$. Suppose that $\mathbf{f}_k(x) = T_J v$ for $x \in I_J$. Let $x \in I_{(0,J)} = \phi_0(I_J)$; then $\phi_0^{-1}(x) = 2x \in I_J$ and

$$\mathbf{f}_{k+1}(x) = \mathbf{T}(\mathbf{f}_k(x)) = T_0 \cdot \mathbf{f}_k(2x) = T_0 T_J v = T_{(0,J)} v.$$

Similarly, if $x \in I_{(1,J)}$, then $\mathbf{f}_{k+1}(x) = T_{(1,J)} v$.

Let $\mathbf{f} = \mathbf{\Phi}(f)$; then $\mathbf{f} = \mathbf{T}\mathbf{f}$ and $\mathbf{f}(x) = T_J \mathbf{f}(\phi_J^{-1}(x))$ for $x \in I_J$. Integrating this over the interval I_J, we obtain

$$[f_{I_J}, \ldots, f_{I_J+N-1}]^t = T_J v.$$

The fact that $\mathbf{f}_k \longrightarrow \mathbf{f}$ in $L^p([0, 1], \mathbf{R}^N)$ follows by a proposition in [R, p. 129]. □

LEMMA 2.5. *Let* v *be a* 2*-eigenvector of* $(T_0 + T_1)$, *and let* $\mathbf{f}_k$ *be defined as above; then for each* k,

$$\int_{[0,1]} \mathbf{f}_k(x) dx = v. \tag{2.3}$$

Proof. Equation (2.3) follows from the following induction argument:

$$\begin{aligned}
\int_{[0,1]} \mathbf{f}_{k+1}(x) dx &= \int_{[0,\frac{1}{2}]} T_0 \cdot \mathbf{f}_k(2x) dx + \int_{[\frac{1}{2},1]} T_1 \cdot \mathbf{f}_k(2x-1) dx \\
&= \frac{1}{2}\left(T_0 \int_{[0,1]} \mathbf{f}_k(x) dx + T_1 \int_{[0,1]} \mathbf{f}_k(x) dx\right) \\
&= \frac{1}{2}(T_0 + T_1) \int_{[0,1]} \mathbf{f}_k(x) dx \\
&= \frac{1}{2}(T_0 + T_1) v = v. \qquad \square
\end{aligned}$$

For any 2-eigenvector v of $(T_0 + T_1)$, we have

$$(T_0 - I)v = -(T_1 - I)v.$$

Let $\tilde{v} = (T_0 - I)v$ and $H(\tilde{v})$ be the subspace in $\mathbf{R}^N$ spanned by

$$\{T_J\tilde{v} : J \in \Lambda\}.$$

THEOREM 2.6. *For* $1 \le p < \infty$, *the following are equivalent:* (i) *equation* (2.1) *has a nonzero* L^p_c*-solution*; (ii) *there exists a* 2*-eigenvector* v *of* $(T_0 + T_1)$ *satisfying*

$$\lim_{l \longrightarrow \infty} \frac{1}{2^l} \sum_{|J|=l} ||T_J\tilde{v}||^p = 0;$$

(iii) *there exists a* 2*-eigenvector* v *of* $(T_0 + T_1)$ *such that there exists an integer* $l \ge 1$ *such that*

$$\frac{1}{2^l} \sum_{|J|=l} ||T_J u||^p < 1 \quad \textit{for all } u \in H(\tilde{v}), \quad ||u|| \le 1. \tag{2.4}$$

Proof. Let $\mathbf{f}_0 = v$ and $\mathbf{f}_{n+1} = \mathbf{T}\mathbf{f}_n$. By Lemma 2.4, for $x \in I_J$ and $|J| = n$, $\mathbf{f}_n(x) = T_J v$. Let $\mathbf{g}_n = \mathbf{f}_{n+1} - \mathbf{f}_n$; then $\mathbf{f}_{n+1} = \mathbf{f}_0 + \mathbf{g}_0 + \cdots + \mathbf{g}_n$, where

$$\mathbf{g}_n(x) = \begin{cases} T_{(J,0)}v - T_J v = T_J\tilde{v} & \text{if } x \in I_{(J,0)}, \\ T_{(J,1)}v - T_J v = -T_J\tilde{v} & \text{if } x \in I_{(J,1)}, \end{cases}$$

and

$$||\mathbf{g}_n||^p = \frac{1}{2^n} \sum_{|J|=n} ||T_J\tilde{v}||^p.$$

Since (i) implies that $||\mathbf{g}_n||$ converges to zero, (ii) follows immediately.

To prove that (ii) implies (iii), we note that $H(\tilde{v})$ is finite dimensional and has a finite basis of $T_{J'}\tilde{v}$'s. Let $u = T_{J'}\tilde{v}$ with $|J'| = k$; then

$$\frac{1}{2^n} \sum_{|J|=n} ||T_J u||^p = \frac{1}{2^n} \sum_{|J|=n} ||T_J T_{J'}\tilde{v}||^p \le 2^k \frac{1}{2^{n+k}} \sum_{|J|=n+k} ||T_J\tilde{v}||^p \longrightarrow 0$$

as $n \longrightarrow \infty$, and the convergence is uniform for all $||u|| \le 1$. Hence (2.4) follows by taking $l = n$ for n sufficiently large.

Now assume (iii) holds. Since $H(\tilde{v})$ is finite dimensional, there is a constant $0 < c < 1$ such that for any $u \in H(\tilde{v})$,

$$\frac{1}{2^l} \sum_{|J|=l} ||T_J u||^p < c||u||^p.$$

For any $|J'| = n$, let $u = T_{J'}\tilde{v} \in H(\tilde{v})$; then

$$\frac{1}{2^l} \sum_{|J|=l} ||T_J T_{J'}\tilde{v}||^p < c||T_{J'}\tilde{v}||^p.$$

Summing over all $|J'| = n$, we have

$$\frac{1}{2^{l+n}} \sum_{|J|=l+n} ||T_J\tilde{v}||^p = \frac{1}{2^{l+n}} \sum_{|J|=l} \sum_{|J'|=n} ||T_J T_{J'}\tilde{v}||^p < \frac{c}{2^n} \sum_{|J'|=n} ||T_{J'}\tilde{v}||^p.$$

It follows from the expression of $||\mathbf{g}_n||$ given above that

$$||\mathbf{g}_{n+l}||^p < c||\mathbf{g}_n||^p.$$

For each fixed n, $\{||\mathbf{g}_{n+kl}||\}_{k=1}^{\infty}$ is dominated by a geometric series, hence $\mathbf{f}_{n+1} = \mathbf{f}_0 + \mathbf{g}_0 + \cdots + \mathbf{g}_n$ converges in L^p. The limit $\mathbf{f}$ is nonzero by Lemma 2.5, and so by Proposition 2.1, (i) follows. □

Remark 1. If

$$\frac{1}{2^l}\sum_{|J|=l} ||T_J|_{H(\tilde{v})}||^p < 1,$$

then (2.4) is satisfied. Hence, if the joint spectral radius [BW1], [DL1] or the mean spectral radius [W] of $\{T_0|_{H(\tilde{v})}, T_1|_{H(\tilde{v})}\}$ is less than 1, then a nonzero L^1-solution exists.

Remark 2. If condition (2.4) is satisfied for one particular norm on $\mathbf{R}^N$, then it will be satisfied for all the (equivalent) norms (the integer l will depend on the choice of norms). This follows directly from Theorem 2.6 (ii).

Also, condition (2.4) can be replaced by the following slightly simpler condition:

$$\frac{1}{2^l}\sum_{|J|=l} ||T_J u_i||^p < 1;$$

where $\{u_1, \ldots, u_k\}$ is a basis of $H(\tilde{v})$. To see this, we define a norm on $\mathbf{R}^N$ such that its restriction on $H(\tilde{v})$ is the l^p-norm given by

$$|||u|||^p = \sum_{i=1}^{k} |\alpha_i|^p, \quad \text{where } u = \sum_{i=1}^{k} \alpha_i u_i.$$

Let $u = \sum_{i=1}^{k} \alpha_i u_i \in H(\tilde{v})$; then

$$\frac{1}{2^l}\sum_{|J|=l} |||T_J u|||^p \le \frac{1}{2^l}\sum_{|J|=l}\sum_{i=1}^{k} |\alpha_i|^p |||T_J u_i|||^p < \sum_{i=1}^{k} |\alpha_i|^p = |||u|||^p,$$

which implies (2.4).

For computational purposes it would be interesting to know the optimal choice of a bound of the integer l in condition (2.4), in particular, when the norm on $\mathbf{R}^N$ is the l^p-norm.

Remark 3. In [DL1, Thm. 3.1 and Rem. 1], it is proved that if $\sum c_n = 2$ and a nonzero compactly supported tempered distributional solution f exists, then the Fourier transform of f must have the form

$$\hat{f}(\xi) = A\prod_{k=1}^{\infty} m_0(2^{-k}\xi),$$

where $m_0(\xi) = \frac{1}{2}\sum_{n=0}^{N-1} c_n e^{in\xi}$. Moreover, if f is integrable, then $A = \int f(x)dx$. It follows that f is unique up to a multiplicative constant. By Proposition 2.3, the above v equals $[f_{[0,1]}, \ldots, f_{[N-1,N]}]^t$, so that the 2-eigenvector satisfying (2.4) is unique. Also, it follows from the expression of A that

$$\sum_{n=0}^{N} v_i = \int f(x)dx \neq 0,$$

hence $v \notin H = \{u : \sum_{i=0}^N u_i = 0\}$.

We are not able to prove these two facts without using the Fourier transform. Nevertheless, we have the following result, whose negation is useful in proving the nonexistence of solutions.

COROLLARY 2.7. *Under the same hypotheses of Theorem 2.6, assume that the solution f exists; then $v \notin H(\tilde{v})$, and the dimension of $H(\tilde{v})$ is $\leq N-1$.*

Proof. By Theorem 2.6 (ii),

$$\frac{1}{2^n}\sum_{|J|=n} ||T_J u||^p \longrightarrow 0 \quad \text{for any } u \in H(\tilde{v}).$$

It follows that if $v \in H(\tilde{v})$, then

$$||v||^p = \frac{1}{2^n}||(T_0+T_1)^n v||^p \leq \frac{1}{2^n}\sum_{|J|=n} ||T_J v||^p \longrightarrow 0$$

as $n \longrightarrow \infty$. This contradicts $v \neq 0$. □

Remark 4. In the construction of the solution f, if we start the iteration from a vector other than the 2-eigenvector v, then the process may not converge, or may converge to the zero function. For example, consider

$$f(x) = f(2x) + f(2x-2).$$

In this case $c_0 = 1, c_1 = 0, c_2 = 1$, and

$$T_0 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad T_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

The 2-eigenvector of $(T_0 + T_1)$ is $v = [1,1]^t$ and $H(\tilde{v}) = 0$, hence condition (2.4) is satisfied. The (normalized) solution f is the characteristic function of the interval $[0, 2]$. However, if we start with the vector $[1,0]^t$, then the iteration with $\mathbf{T}$ will not converge.

Nevertheless, we are still able to choose a large class of vectors that can serve as initial values.

COROLLARY 2.8. *Suppose $\sum c_n = 2$. Let $w = [w_1, w_2, \ldots, w_N]^t$ be a vector in $\mathbf{R}^N$ and $H'(w)$ be the subspace spanned by*

$$\{T_J(T_i - I)w : \quad J \in \Lambda, \quad i = 0,1\}.$$

Suppose $w \notin H'(w)$ or $\sum w_i \neq 0$, and suppose

$$\frac{1}{2^l}\sum_{|J|=l} ||T_J u||^p < 1 \quad \textit{for all } u \in H'(w), \quad ||u|| \leq 1;$$

then (2.1) *has a nonzero L_c^p-solution.*

Proof. As in the first part of the proof of Theorem 2.6, we define $\mathbf{f}_0(x) = w$ for all $x \in [0,1]$ and $\mathbf{f}_{k+1} = \mathbf{T}\mathbf{f}_k$; then $\mathbf{f}_k$ will converge in $L^p([0,1], \mathbf{R}^N)$ and the limiting function $\mathbf{f}$ will satisfy equation (2.2).

We still need to show that $\mathbf{f} \neq 0$. If $w \notin H'(w)$, then from the proof of Theorem 2.6 we have

$$\mathbf{f} = \lim_{n\to\infty} \mathbf{f}_k = \mathbf{f}_0 + \lim_{n\to\infty} \sum_{j=0}^{n} \mathbf{g}_j$$

with $\mathbf{f}_0(x) \notin H'(w)$ and $\mathbf{g}_n(x) \in H'(w)$, so $\mathbf{f}(x) \notin H'(w)$ and $\mathbf{f}$ is nonzero.

To prove the second case, we assume that $\sum w_i \neq 0$; then by Proposition 2.2, $e = [1, \dots, 1]$ is a left 2-eigenvector of $(T_0 + T_1)$, so

$$\begin{aligned} &e \cdot \int_{[0,1]} \mathbf{f}_{k+1}(x)dx \\ &= e \cdot \int_{[0,\frac{1}{2}]} T_0 \cdot \mathbf{f}_k(2x)dx + e \cdot \int_{[\frac{1}{2},1]} T_1 \cdot \mathbf{f}_k(2x-1)dx \\ &= \frac{1}{2} e \cdot \left(T_0 \int_{[0,1]} \mathbf{f}_k(x)dx + T_1 \int_{[0,1]} \mathbf{f}_k(x)dx \right) \\ &= \frac{1}{2} e \cdot (T_0 + T_1) \int_{[0,1]} \mathbf{f}_k(x)dx = e \cdot \int_{[0,1]} \mathbf{f}_k(x)dx. \end{aligned}$$

Repeating this argument, we have

$$e \cdot \int_{[0,1]} \mathbf{f}_{k+1}(x)dx = \cdots = e \cdot \mathbf{f}_0(x) = e \cdot w = \sum w_i \neq 0.$$

This implies that $\int_{[0,1]} \mathbf{f} \neq 0$, and the proof is complete. □

Remark 5. Let D_l^p be the set of $(c_0, \dots, c_N)$ for which (2.4) holds; then
(i) $D_l^p \subseteq D_{2l}^p$, and
(ii) $D_l^p \subseteq \cup_k D_{2^k}^p$ for any l.
Indeed, if (2.4) holds for some l, then

$$\frac{1}{2^l} \sum_{|J|=l} ||T_J u||^p < ||u||^p \quad \text{for all} \quad u \in H(\tilde{v}).$$

Since $T_J u \in H(\tilde{v})$ if $u \in H(\tilde{v})$, we have

$$\frac{1}{2^{2l}} \sum_{|J|=2l} ||T_J u||^p \leq \frac{1}{2^l}\frac{1}{2^l} \sum_{|J'|=l} \sum_{|J''|=l} ||T_{J'} T_{J''} u||^p < \frac{1}{2^l} \sum_{|J''|=l} ||T_{J''} u||^p \leq ||u||^p.$$

To show (ii), let c be a number $0 < c < 1$ such that

$$\frac{1}{2^l} \sum_{|J|=l} ||T_J u||^p < c||u||^p \quad \text{for all} \quad u \in H(\tilde{v})$$

holds. If $|J| = 2^k$, we write $J = (J_1, \dots, J_m, J')$, where $|J_i| = l$ and $|J'| < l$; then

$$\begin{aligned} \frac{1}{2^{2^k}} \sum_{|J|=2^k} ||T_J u||^p &\leq \frac{1}{2^{|J'|}} \left(\frac{1}{2^l}\right)^m \sum_{J_1=l} \cdots \sum_{J_m=l} \sum_{J'} ||T_{J_1} \cdots T_{J_m} T_{J'} u||^p \\ &< c^m \frac{1}{2^{|J'|}} \sum_{J'} ||T_{J'} u||^p, \end{aligned}$$

which is less than 1 for $||u|| \leq 1$ if k (hence m) is large enough.

COROLLARY 2.9. *Equation* (2.1) *has a nonzero L^p_c-solution if and only if* $(c_0, \ldots, c_N) \in \cup_{k=1}^{\infty} D^p_{2^k}$.

Remark 6. One can also consider the functional equation

$$f(x) = \sum c_n f(kx - \beta n) \tag{2.5}$$

for some integer $k > 1$ and constant $\beta \neq 0$. Note that the L^p_c-solution will be supported by $[0, \beta N/(k-1)]$ [DL1]. Theorem 2.6 still holds with minor modifications of the proof. The matrices for the cascade algorithm will be

$$T_m = [c_{ki+m-j}], \quad 0 \leq i, j \leq N-1,$$

for $m = 0, \ldots, k-1$. If we define $\phi_m(x) = \frac{x}{k} + \frac{m\beta}{k}$, $m = 0, \ldots, k-1$, the vector form of equation (2.5) becomes

$$\mathbf{f}(x) = \sum_{m=0}^{k-1} T_m \mathbf{f}(\phi_m^{-1}(x)),$$

and the proof follows as above.

THEOREM 2.10. *Suppose* $\sum_{n=0}^{N} c_n = k$ *and* $1 \leq p < \infty$. *Then equation* (2.5) *has a nonzero L^p_c-solution if and only if there exists a k-eigenvector v of $\sum_m T_m$ satisfying the following: there exists an integer $l \geq 1$ such that*

$$\frac{1}{k^l} \sum_{j_i=0,\ldots,k-1} ||T_{j_1} \cdots T_{j_l} u||^p < 1$$

for all vectors u, with $||u|| \leq 1$, in the smallest subspace containing $(T_m - I)v$ and which is invariant under T_m ,$m = 0, \ldots, k-1$.

To conclude this section we consider some special cases of Theorem 2.6. First, we assume that $\sum c_{2n} = \sum c_{2n+1} = 1$. This is a necessary condition for the solution to be the scaling function of a wavelet that defines a multiresolution (see [DL1], [CH2]). By Proposition 2.2, we know that $e = [1, \ldots, 1]$ is a common left 1-eigenvector of the two matrices T_0 and T_1. Since $\mathbf{f}_n(x) = T_J v$ if $x \in I_J = \phi_J([0, 1))$,

$$e \cdot \mathbf{f}_n(x) = e \cdot T_J v = e \cdot v = \sum v_i.$$

Hence $e \cdot \mathbf{f}(x)$ equals the constant $\sum v_i$ for almost all $x \in [0, 1]$; that is,

$$\sum_{n=0}^{N-1} f(x+n) = \sum v_i \quad \text{for almost all } x \in [0, 1],$$

and

$$\int_0^N f(x)dx = \sum_{n=0}^{N-1} \int_0^1 f(x+n)dx = \sum v_i,$$

which is not zero as we mentioned in Remark 3.

Let H be the hyperplane of $\mathbf{R}^N$ defined by

$$H = \left\{ [u_0, \ldots, u_{N-1}]^t : \quad \sum u_j = 0 \right\};$$

then H is invariant under T_0 and T_1. For any vector $v \in \mathbf{R}^N$, we always have $(T_0 - I)v$, $(T_1 - I)v \in H$ (since the sum of the coordinates of $(T_0 - I)v$ equals $e \cdot (T_0 - I)v = 0$). Hence $H(\tilde{v}) \subseteq H$.

COROLLARY 2.11. *Suppose that* $\sum c_{2n} = \sum c_{2n+1} = 1$. *If there exists an integer* $l \geq 1$ *such that*

$$\frac{1}{2^l} \sum_{|J|=l} ||T_J u||^p < 1 \quad \textit{for all } u \in H, \quad ||u|| \leq 1, \tag{2.6}$$

then equation (2.1) *has nonzero* L^p_c*-solutions.*

Assuming $\sum c_i = 2$, the condition $\sum c_{2n} = \sum c_{2n+1} = 1$ is equivalent to $\sum (-1)^n c_n = 0$. More generally, we can consider the *m-sum rules*; that is,

$$\sum_{n=0}^{N} (-1)^n n^j c_n = 0 \quad \text{for } j = 0, 1, \ldots, m. \tag{2.7}$$

The m-sum rule is used to ensure higher order of regularity (see [DL2]). It is known that there is a matrix B such that

$$BT_0B^{-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ * & 1/2 & & \vdots & \vdots & & \vdots \\ \vdots & & & 0 & 0 & & 0 \\ * & \cdots & * & 1/2^m & 0 & \cdots & 0 \\ * & * & \cdots & * & * & \cdots & * \\ \vdots & & & \vdots & \vdots & & \vdots \\ * & * & \cdots & * & * & \cdots & * \end{bmatrix},$$

$$BT_1B^{-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ * & 1/2 & & \vdots & \vdots & & \vdots \\ \vdots & & & 0 & 0 & & 0 \\ * & \cdots & * & 1/2^m & 0 & \cdots & 0 \\ * & * & \cdots & * & * & \cdots & * \\ \vdots & & & \vdots & \vdots & & \vdots \\ * & * & \cdots & * & * & \cdots & * \end{bmatrix}.$$

The matrix B can be orthonormalized by the Gram–Schmidt process; the first row of B is the vector $[1/\sqrt{N}, \ldots, 1/\sqrt{N}]$ and the first $(m+1)$ rows of B are linear combinations of vectors

$$[1^j, 2^j, \ldots, N^j], \qquad j = 0, 1, \ldots, m.$$

Let $H' = \{[0, u_1, \ldots, u_{N-1}]^t\}$, $H'_m \subseteq H'$ be the subspace of vectors whose first $(m+1)$ components are zero. Then $H = B^{-1}H'$, and $H_m := B^{-1}H'_m$ is actually the following subspace:

$$H_m = \left\{ [u_0, \ldots, u_{N-1}]^t : \quad \sum_{n=0}^{N-1} n^j u_n = 0, j = 0, 1, \ldots, m \right\}$$

(here $0^0 = 1$).

COROLLARY 2.12. *Suppose $\sum c_n = 2$ and the m-sum rules hold. If there exists an integer $l \geq 1$ such that*

$$\frac{1}{2^l} \sum_{|J|=l} ||T_J u||^p < 1 \quad \textit{for all } u \in H_m, \quad ||u|| \leq 1,$$

then equation (2.1) has a nonzero L^p_c-solution.

Proof. For $j = 1, \dots, m$, let $v'_j = [0, \dots, 0, 1, 0, \dots, 0]^t$ be the vectors whose jth component is 1. Let $v_j = B^{-1}v'_j$; then $v'_j \in H'$, $v_j \in H$, and

$$BT_0 v_j = BT_0 B^{-1} \cdot Bv_j = BT_0 B^{-1} v'_j = \frac{1}{2^j} v'_j + w'_j$$

for some $w'_j \in H'_m$, so $T_0 v_j = (1/2^j)v_j + w_{0,j}$ for some $w_{0,j} \in H_m$. Similarly, $T_1 v_j = (1/2^j)v_j + w_{1,j}$ for some $w_{1,j} \in H_m$.

Note that $\{v_1, \dots, v_m\}$ and H_m span H. By Corollary 2.11 and Remark 2, we need only show that for each $j = 1, \dots, m$ there exists k such that

$$s_k := \frac{1}{2^k} \sum_{|J|=k} ||T_J v_j||^p < 1.$$

Let $c_p = 2^{p-1}$; then for the usual l^p norm we have $||u+v||^p \leq c_p(||u||^p + ||v||^p)$ for any vectors u and v. For any $j = 1, \dots, m$ and $\eta = 1/4c_p$, by assumption there is an integer l such that

$$\frac{1}{2^l} \sum_{|J|=l} ||T_J w_{i,j}||^p < \eta \quad \text{for } i = 0, 1.$$

Then

$$\begin{aligned} s_n &= \frac{1}{2^n}\left[\sum_{|J|=n-1} \left\|T_J\left(\frac{1}{2^j}v_j + w_{0,j}\right)\right\|^p + \sum_{|J|=n-1} \left\|T_J\left(\frac{1}{2^j}v_j + w_{1,j}\right)\right\|^p\right] \\ &\leq \frac{1}{2^n}\left[\frac{2c_p}{2^{jp}} \sum_{|J|=n-1} ||T_J v_j||^p + c_p \sum_{|J|=n-1} ||T_J w_{0,j}||^p + c_p \sum_{|J|=n-1} ||T_J w_{1,j}||^p\right] \\ &< \frac{c_p}{2^{jp}} s_{n-1} + \eta c_p \leq \frac{1}{2} s_{n-1} + \frac{1}{4}. \end{aligned}$$

Hence

$$s_{n+l} < \frac{1}{2^n} s_l + \left(\frac{1}{2^{n-1}} + \frac{1}{2^{n-2}} + \dots + 1\right)\frac{1}{4} \leq \frac{1}{2} + \frac{1}{2} = 1$$

for sufficiently large n. □

3. Special cases: $N \leq 3$. The simplest nontrivial 2-dilation equation occurs when $N = 2$, i.e.,

$$f(x) = c_0 f(2x) + c_1 f(2x-1) + c_2 f(2x-2), \tag{3.1}$$

where $c_0 + c_1 + c_2 = 2$.

THEOREM 3.1. *For $1 \le p < \infty$, equation* (3.1) *has a* (*nonzero*) L_c^p*-solution if and only if either* $c_1 = 1$ *and*

$$\frac{1}{2}(|c_0|^p + |1 - c_0|^p) < 1,$$

or $c_0 = c_2 = 1$. *In the later case* $f = c\chi_{[0,2)}$.

Proof. We will use the l^p-norm on $\mathbf{R}^2$. Note that

$$T_0 = \begin{bmatrix} c_0 & 0 \\ c_2 & c_1 \end{bmatrix}, \quad T_1 = \begin{bmatrix} c_1 & c_0 \\ 0 & c_2 \end{bmatrix}, \quad \text{and } T_0 + T_1 = \begin{bmatrix} c_0 + c_1 & c_0 \\ c_2 & c_1 + c_2 \end{bmatrix}.$$

If $(c_0, c_2) = (0,0)$, then $(T_0 + T_1) = 2I$. Any nonzero vector $v = [x,y]^t$ will be a 2-eigenvector. It is a direct calculation that $v \in H(\tilde{v})$ and, by Corollary 2.7, no nonzero L_c^p-solution exists.

We assume that $(c_0, c_2) \neq (0,0)$; the 2-eigenvector of $(T_0 + T_1)$ is $v = [c_0, c_2]^t$, so that

$$\tilde{v} = (T_0 - I)v = \begin{bmatrix} c_0(c_0 - 1) \\ c_2(1 - c_2) \end{bmatrix}. \tag{3.2}$$

For an L_c^p-solution to exist, $H(\tilde{v})$ can only be $\{0\}$ or one-dimensional (Corollary 2.7).

In the first case, $\tilde{v} = 0$, condition (2.4) is automatically satisfied. The only possible cases are

$$(c_0, c_2) = (1,1), \ (0,1), \ \text{or } (1,0),$$

and the (normalized) solutions are given by $f(x) = \chi_{[0,2)}, \ \chi_{[1,2)}$, or $\chi_{[0,1)}$, respectively.

In the second case, $\tilde{v} \neq 0$. Since $H(\tilde{v})$ is invariant under T_0 and T_1, $T_0\tilde{v} = c\tilde{v}$ for some c. Expression (3.2) yields the following cases (excluding those considered above):

(a) $c_i = 0$ for $i = 0$ or 2. In this case $v \in H(\tilde{v})$ and Corollary 2.7 implies that (3.1) has no L_c^p-solution.

(b) $c_i = 1$ for $i = 0$ or 2. In this case a direct calculation shows that $T_0\tilde{v}$, $T_1\tilde{v}$ are independent. Hence $H(\tilde{v})$ is two-dimensional and by Corollary 2.7 no L_c^p-solution exists.

(c) $c_i \neq 0, 1$ for $i = 0$ and 2. By equating (3.2) and

$$T_0\tilde{v} = \begin{bmatrix} c_0^2(c_0 - 1) \\ c_2(c_0^2 + c_2^2 + c_0c_2 - 2c_0 - 3c_2 + 2) \end{bmatrix} \tag{3.3}$$

with $T_0\tilde{v} = c\tilde{v}$, we have $c = c_0$, so that by (3.2) and (3.3),

$$(c_0^2 + c_2^2 + c_0c_2 - 2c_0 - 3c_2 + 2) = c_0(1 - c_2);$$

that is

$$(c_0 + c_2 - 2)(c_0 + c_2 - 1) = 0.$$

Hence, either (i) or (ii) below holds.

(i) $c_0 + c_2 = 2$. In this case $v = [c_0, 2 - c_0]^t$ and $\tilde{v} = (c_0 - 1)v$. Once again $v \in H(\tilde{v})$ and no L_c^p-solution exists.

(ii) $c_0 + c_2 = 1$. In this case a direct calculation shows that $T_0\tilde{v} = c_0\tilde{v}$, $T_1\tilde{v} = c_2\tilde{v}$. By Theorem 2.6, equation (3.1) has an L_c^p-solution if and only if there exists an integer $l \geq 1$ such that

$$\frac{1}{2^l}(|c_0|^p + |c_2|^p)^l||\tilde{v}||^p = \frac{1}{2^l}\sum_{|J|=l}||T_J\tilde{v}||^p < ||\tilde{v}||^p.$$

This is equivalent to

$$\frac{1}{2}(|c_0|^p + |1-c_0|^p) < 1.$$

The theorem follows by summarizing all the cases. □

It follows directly from the theorem that if $c_0 + c_2 = 1$ and if

(a) $c_0 \in (-\frac{1}{2}, \frac{3}{2})$, then an L_c^1-solution exists;

(b) $c_0 \in (\frac{1-\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2})$, then an L_c^2-solution exists;

(c) $c_0 \in (0,1)$, then an L_c^p-solution exists for all $1 \leq p < \infty$.

The conditions are also necessary except for $f = \chi_{[0,2)}$. We remark that in [W] it is proved that if $c_0 + c_2 = 1$, then equation (3.1) has a continuous solution if and only if $c_0 \in (0,1)$, which is stronger than (c). Other proofs of the L^1-, L^2-cases in (a) and (b) are also known (see [P]).

We will now consider the 2-dilation equation with $N = 3$:

$$f(x) = c_0 f(2x) + c_1 f(2x-1) + c_2 f(2x-2) + c_3 f(2x-3) \tag{3.4}$$

with the stronger assumption $c_0 + c_2 = c_1 + c_3 = 1$. The matrices T_0 and T_1 are given by

$$T_0 = \begin{bmatrix} c_0 & 0 & 0 \\ c_2 & c_1 & c_0 \\ 0 & c_3 & c_2 \end{bmatrix}, \quad T_1 = \begin{bmatrix} c_1 & c_0 & 0 \\ c_3 & c_2 & c_1 \\ 0 & 0 & c_3 \end{bmatrix}.$$

It is easy to show that $(T_0 + T_1)$ has 1, 2, and $(1 - c_0 - c_3)$ as eigenvalues, and the 2-eigenvector is

$$v = \begin{bmatrix} c_0(1+c_0-c_3) \\ (1+c_0-c_3)(1-c_0+c_3) \\ c_3(1-c_0+c_3) \end{bmatrix}$$

provided that $(c_0, c_3) \neq (0,-1)$ or $(-1,0)$ (in these cases the 2-eigenvectors are given by $[1,0,0]^t$ and $[0,0,1]^t$, respectively). It follows that (excluding the two exceptional cases),

$$\tilde{v} = (T_0 - I)v = \begin{bmatrix} c_0(c_0-1)(1+c_0-c_3) \\ -c_0(c_0-1)(1+c_0-c_3) - c_3(1-c_3)(1-c_0+c_3) \\ c_3(1-c_3)(1-c_0+c_3) \end{bmatrix}.$$

Recall that the subspace $H(\tilde{v})$ is generated by $T_J(\tilde{v})$, $J \in \Lambda$. Under the assumption $c_0 + c_2 = c_1 + c_3 = 1$, T_J is invariant on

$$H = \{[x,y,z]^t : x + y + z = 0\},$$

and $H(\tilde{v}) \subseteq H$. For convenience we will reduce the matrices T_0 and T_1 on H by considering the first and the third coordinates of $[x,y,z]^t$ in H. This defines two matrices S_i, $i = 0,1$, as in [CH1]. This can be seen by the following diagram:

$$\begin{array}{ccc} H & \xrightarrow{T_i} & H \\ \tau \uparrow & & \downarrow \tau^{-1}, \\ \mathbf{R}^2 & \xrightarrow{S_i} & \mathbf{R}^2 \end{array}$$

where $S_i = \tau^{-1}T_i\tau$ with $\tau : \mathbf{R}^2 \longrightarrow H$ denoting the natural isomorphism $[x, z] \longrightarrow [x, -(x+z), z]$. The explicit expression of S_i, $i = 1, 2$, is given by

$$S_0 = \begin{bmatrix} c_0 & 0 \\ -c_3 & 1-c_0-c_3 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 1-c_0-c_3 & -c_0 \\ 0 & c_3 \end{bmatrix}.$$

Slightly abusing the notation, we will still use $\tilde{v}$ and $H(\tilde{v})$ in $\mathbf{R}^2$ for the corresponding terms in H. The new $\tilde{v}$ in $\mathbf{R}^2$ is given by

$$\tilde{v} = \begin{bmatrix} c_0(c_0-1)(1+c_0-c_3) \\ c_3(1-c_3)(1-c_0+c_3) \end{bmatrix} \tag{3.5}$$

for $(c_0, c_3) \neq (0, -1)$ or $(-1, 0)$. The following theorem follows readily from Theorem 2.6.

THEOREM 3.2. *For $1 \le p < \infty$, equation (3.4) has a nonzero L^p_c-solution if and only if there exists an integer l such that*

$$\frac{1}{2^l} \sum_{|J|=l} ||S_J u||^p < 1 \quad \textit{for all } u \in H(\tilde{v}) \quad \textit{and } ||u|| \le 1. \tag{3.6}$$

For the degenerate case (i.e., $H(\tilde{v}) = \{0\}$ or one-dimensional), condition (3.6) can be displayed explicitly. This is shown in the following two lemmas.

LEMMA 3.3. $H(\tilde{v}) = \{0\}$ *if and only if* $(c_0, c_3) \in \{(0,0), (1,0), (0,1), (1,1)\}$.

Proof. This is a consequence of (3.5), and a direct computation of the two special cases $(c_0, c_3) = (0, -1)$ or $(-1, 0)$ (for such cases the corresponding $H(\tilde{v})$ are two dimensional). □

The solutions for these special cases can be handled easily as follows:

If $(c_0, c_3) = (0, 0)$, then the solutions are $f = c\chi_{[1,2]}$.
If $(c_0, c_3) = (1, 0)$, then the solutions are $f = c\chi_{[0,1]}$.
If $(c_0, c_3) = (0, 1)$, then the solutions are $f = c\chi_{[2,3]}$.
If $(c_0, c_3) = (1, 1)$, then the solutions are $f = c\chi_{[0,3]}$.

It is also simple to show that for the exceptional cases $(c_0, c_3) = (0, -1)$ or $(-1, 0)$, condition (3.6) is not satisfied; therefore there is no L^p_c-solution.

LEMMA 3.4. $H(\tilde{v})$ *is one-dimensional if and only if* $(c_0, c_3) \notin \{(0,0), (1,0), (0,1), (1,1)\}$ *and one of the following holds*:

$$c_0 = 0, \quad c_3 = 0, \quad \textit{or } 1 - c_0 - c_3 = 0.$$

Let $\tilde{v}_0 = S_0\tilde{v}$, $\tilde{v}_1 = S_1\tilde{v}$. *Then for the above three cases we have*

$$\tilde{v} = c[0,1]^t, \quad \textit{and } \tilde{v}_0 = (1-c_0-c_3)\tilde{v}, \; \tilde{v}_1 = c_3\tilde{v};$$
$$\tilde{v} = c[1,0]^t, \quad \textit{and } \tilde{v}_0 = c_0\tilde{v}, \; \tilde{v}_1 = (1-c_0-c_3)\tilde{v};$$
$$\tilde{v} = c[c_0, -c_3]^t, \quad \textit{and } \tilde{v}_0 = c_0\tilde{v}, \; \tilde{v}_1 = c_3\tilde{v},$$

respectively.

Proof. The sufficiency is clear; we only prove the necessity. Assuming c_0, $c_3 \neq 0$, we want to show that $(1 - c_0 - c_3) = 0$.

Suppose $(1 - c_0 - c_3) \neq 0$. Note that S_0 has two eigenvalues c_0 and $(1 - c_0 - c_3)$ with corresponding eigenvectors $[1-2c_0-c_3, c_3]^t$ and $[0, 1]^t$, and S_1 has two eigenvalues c_3 and $(1 - c_0 - c_3)$ with corresponding eigenvectors $[c_0, 1 - c_0 - 2c_3]^t$ and $[1, 0]^t$. The one-dimensional assumption implies that $S_0\tilde{v} = \lambda_0\tilde{v}$, $S_1\tilde{v} = \lambda_1\tilde{v}$ for some constants λ_0, λ_1. Then it follows that

$$\lambda_0 = c_0, \quad \lambda_1 = c_3,$$

and

$$\tilde{v} = c' \begin{bmatrix} 1 - 2c_0 - c_3 \\ c_3 \end{bmatrix} = c'' \begin{bmatrix} c_0 \\ 1 - c_0 - 2c_3 \end{bmatrix}$$

for some constants c', c''. Thus

$$(1 - 2c_0 - c_3)(1 - c_0 - 2c_3) = c_0c_3,$$

and we have either $(1 - c_0 - c_3) = 0$ or $(1 - 2c_0 - 2c_3) = 0$. Since $(1 - c_0 - c_3) \neq 0$, we must have $(1 - 2c_0 - 2c_3) = 0$, so $\tilde{v} = c'''[1, 1]^t$. By the formula of $\tilde{v}$ in (3.5), we have

$$c_0(c_0 - 1)(1 + c_0 - c_3) = c_3(1 - c_3)(1 - c_0 + c_3).$$

Simplifying this, we end up with $0 = 3/8$, which is a contradiction. □

THEOREM 3.5. *Let $1 \leq p < \infty$. Suppose that $c_0 + c_2 = c_1 + c_3 = 1$ and one of c_0, c_3, or $1 - c_0 - c_3$ is zero; then equation* (3.4) *has nonzero L^p_c-solutions if and only if*

$$|c_0|^p + |c_3|^p + |1 - c_0 - c_3|^p < 2. \tag{3.7}$$

Proof. Let $||\cdot||$ be the l^p-norm on $\mathbf{R}^2$. In view of Lemmas 3.3 and 3.4, we can assume that $\tilde{v} \neq 0$ and $H(\tilde{v})$ is one-dimensional. We first consider $c_0 = 0$. The fact that $S_0u = (1 - c_3)u$, $S_1u = c_3u$ for any $u \in H(\tilde{v})$ yields

$$\frac{1}{2^l} \sum_{|J|=l} ||S_Ju||^p = \frac{1}{2^l}(|1 - c_3|^p + |c_3|^p)^l.$$

Now apply Theorem 3.2. We see that equation (3.4) has nonzero L^p_c-solutions if and only if $|1 - c_3|^p + |c_3|^p < 2$.

Similarly, we can show that the corresponding conditions for $c_3 = 0$ and $1 - c_0 - c_3 = 0$ are $|c_0|^p + |1 - c_0|^p < 2$ and $|c_0|^p + |c_3|^p < 2$, respectively. This completes the proof. □

The following is an improvement of Theorem 3.2.

THEOREM 3.6. *For $1 \leq p < \infty$, equation* (3.4) *has a nonzero L^p_c-solution if and only if either $(c_0, c_3) = (1, 1)$ or there exists an integer l such that*

$$\frac{1}{2^l} \sum_{|J|=l} ||S_Ju||^p < 1 \quad \textit{for all } u \in \mathbf{R}^2 \quad \textit{and } ||u|| \leq 1. \tag{3.8}$$

Proof. By Theorem 3.2, we need only show that condition (3.8) holds when $H(\tilde{v})$ is zero or one-dimensional.

The case when $H(\tilde{v}) = \{0\}$ is obvious by Lemma 3.3, so we suppose that $H(\tilde{v})$ is one-dimensional. Equation (3.7) implies that

$$\frac{1}{2}(||S_0u||^p_p + ||S_1u||^p_p) < 1$$

for $u = [0, 1]^t$ and $[1, 0]^t$, which is a basis of $\mathbf{R}^2$; therefore the theorem follows by Remark 2. □

As a special case, we have the following corollary.

COROLLARY 3.7. *Let* $1 \le p < \infty$. *Suppose* $c_0 + c_2 = c_1 + c_3 = 1$ *and*

$$|c_0|^p + |c_3|^p + |1 - c_0 - c_3|^p < 2; \tag{3.9}$$

then (3.4) *has nonzero* L^p_c*-solutions.*

Proof. Let $||.||$ be the l^p-norm. As mentioned in Remark 2, we need only verify condition (3.6) for a basis of $\mathbf{R}^2$. Therefore, condition (3.9) implies (3.8) for $u = [1, 0]^t$ and $[0, 1]^t$ with $l = 1$. □

Similarly, we can take l to be other integers and obtain sufficient conditions for (3.4) to have nonzero L^p_c-solutions. However, the expression is more complicated. For example, for $p = 1$ the condition of (3.8) for $l = 2$ is equivalent to

$$\begin{aligned} &c_0^2 + c_3^2 + (1 - c_0 - c_3)^2 + |c_0(1 - c_0)| + |c_3(1 - c_3)| + |c_0(1 - c_0 - c_3)| \\ &\quad + |c_3(1 - c_0 - c_3)| < 4. \end{aligned} \tag{3.10}$$

In the appendix we will plot the different regions of (c_0, c_3) that admit solutions. They include the ones determined by (3.9) and (3.10), and some other known regions.

4. L^2-solutions. In this section we will show that condition (2.4) in Theorem 2.6 can be reduced to a more explicit form for the case when $p = 2$. We will use the Euclidean norm on $\mathbf{R}^N$. For any $u \in \mathbf{R}^N$, let $A_k(u) := (1/2^k) \sum_{|J|=k} ||T_J u||^2$; then

$$\begin{aligned} A_k(u) &= \frac{1}{2^k} \sum_{|J|=k} ||T_J u||^2 = \frac{1}{2^k} \sum_{|J|=k} u^t T_J^t T_J u \\ &= \frac{1}{2^k} u^t \left(\sum_{|J|=k} T_J^t T_J \right) u = \frac{1}{2^k} u^t M_k u, \end{aligned} \tag{4.1}$$

where $M_k := \sum_{|J|=k} T_J^t T_J$, and M_0 is the identity matrix. Since $T_J = T_{(J',0)}$ or $T_{(J',1)}$ for some J', it is easy to see that M_k satisfies the inductive identity

$$M_{k+1} = T_0^t M_k T_0 + T_1^t M_k T_1.$$

The matrix M_k is actually determined by its first column; its explicit form is given as follows.

LEMMA 4.1. *For any integer* $k \ge 0$, M_k *has the following form*:

$$M_k = [\alpha^{(k)}_{|i-j|}] = \begin{bmatrix} \alpha_0^{(k)} & \alpha_1^{(k)} & \cdots & \alpha_{N-1}^{(k)} \\ \alpha_1^{(k)} & \alpha_0^{(k)} & \cdots & \alpha_{N-2}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{N-1}^{(k)} & \alpha_{N-2}^{(k)} & \cdots & \alpha_0^{(k)} \end{bmatrix}.$$

If we let $\alpha^{(k)} = [\alpha_0^{(k)}, \dots, \alpha_{N-1}^{(k)}]^t$, *then* $\alpha^{(k)} = W\alpha^{(k-1)} = W^k e_1$, *where* $e_1 = [1, 0 \dots, 0]$, *and* W *is an* $N \times N$ *matrix with*

$$(W)_{i,j} = \sum_{m=-\infty}^{\infty} c_{i+m} c_{m \pm 2j}, \quad 0 \le i, j \le N-1,$$

where

$$c_{i\pm 2j} := \begin{cases} c_i & \text{if } j = 0, \\ c_{i+2j} + c_{i-2j} & \text{if } j \neq 0. \end{cases}$$

Proof. We prove the lemma by induction. Supposing M_k has the form as given, let $\alpha_{-l}^{(k)} = \alpha_l^{(k)}$; by using $M_{k+1} = T_0^t M_k T_0 + T_1^t M_k T_1$, the (i,j) entry of M_{k+1} is given by

$$\begin{aligned}
(M_{k+1})_{i,j} &= \sum_{m=1}^{N}\sum_{n=1}^{N} c_{2m-i-1}\alpha_{m-n}^{(k)} c_{2n-j-1} + \sum_{m=1}^{N}\sum_{n=1}^{N} c_{2m-i}\alpha_{m-n}^{(k)} c_{2n-j} \\
&= \sum_{n=1}^{N}\sum_{l=-(N-1)}^{N-1} c_{2n+2l-i-1}c_{2n-j-1}\alpha_l^{(k)} + \sum_{n=1}^{N}\sum_{l=-(N-1)}^{N-1} c_{2n+2l-i}c_{2n-j}\alpha_l^{(k)} \\
&\qquad (l = m-n) \\
&= \sum_{m=1}^{2N}\sum_{l=-(N-1)}^{N-1} c_{m+2l-i}c_{m-j}\alpha_l^{(k)} \qquad (m = 2n \text{ or } m = 2n-1) \\
&= \sum_{l=-(N-1)}^{N-1}\sum_{m=-\infty}^{\infty} c_{m+2l}c_{m-j+i}\alpha_l^{(k)} \\
&= \sum_{l=0}^{N-1}\sum_{m=-\infty}^{\infty} c_{m\pm 2l}c_{m-j+i}\alpha_l^{(k)}.
\end{aligned}$$

(We can extend the sum from $-\infty$ to $+\infty$ since c_n vanishes for $n \notin \{0, \dots N-1\}$.) By the symmetry of the range of l and m, we can rewrite the above equation as

$$(M_{k+1})_{i,j} = \sum_{l=0}^{N-1}\sum_{m=-\infty}^{\infty} c_{m\pm 2l}c_{m-i+j}\alpha_l^{(k)}.$$

Hence $(M_{k+1})_{i,j} = (M_{k+1})_{i+1,j+1} = (M_{k+1})_{j,i}$, and M_k has the form as asserted. Also from the proof above, we see that

$$\alpha_i^{(k+1)} = (M_{k+1})_{i,0} = \sum_{l=0}^{N-1}\sum_{m=-\infty}^{\infty} c_{m\pm 2l}c_{m-i}\alpha_l^{(k)}.$$

Therefore we may write

$$\begin{bmatrix} \alpha_0^{(k+1)} \\ \alpha_1^{(k+1)} \\ \vdots \\ \alpha_{N-1}^{(k+1)} \end{bmatrix} = W \begin{bmatrix} \alpha_0^{(k)} \\ \alpha_1^{(k+1)} \\ \vdots \\ \alpha_{N-1}^{(k)} \end{bmatrix} = W^{k+1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where W is an $N \times N$ matrix with (i,j) entry as

$$(W)_{i,j} = \sum_{m=-\infty}^{\infty} c_{i+m}c_{m\pm 2j}, \quad 0 \le i,j \le N-1. \qquad \square$$

PROPOSITION 4.2. *The matrix W can be written as the product $A \cdot B$, where*

$$A = [c_{i+j}]_{\substack{0\le i\le N-1,\\ |j|\le N-1}}, \qquad B = [c_{i\pm 2j}]_{\substack{|i|\le N-1,\\ 0\le j\le N-1}}.$$

For any vectors $u = [u_0, \ldots, u_{N-1}]^t$, let $\Psi(u)$ be the vector

$$\Psi(u) = \left[\sum u_i^2, 2\sum u_i u_{i+1}, \ldots, 2\sum u_i u_{i+N-1}\right]^t;$$

then Theorem 2.6 can be written as follows.

PROPOSITION 4.3. *Equation (2.1) has a nonzero L_c^2-solution if and only if there is a 2-eigenvector v of $(T_0 + T_1)$ such that for any $u \in H(\tilde{v})$,*

$$\lim_{l\to\infty} \frac{1}{2^l}\Psi(u)^t \cdot W^l \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0. \tag{4.2}$$

Proof. For any $u = [u_0, \ldots, u_{N-1}]^t \in H(\tilde{v})$, by (4.1) and Lemma 4.1, we have

$$\begin{aligned} A_l(u) &= \frac{1}{2^l}\sum_{|J|=l} ||T_J u||^2 = \frac{1}{2^l} u^t M_l u \\ &= \frac{1}{2^l}[u_0, \ldots, u_{N-1}] M_l \begin{bmatrix} u_0 \\ \vdots \\ u_{N-1} \end{bmatrix} = \frac{1}{2^l}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} u_i u_j \alpha_{|i-j|}^{(l)} \\ &= \frac{1}{2^l}\Psi(u)^t \cdot \begin{bmatrix} \alpha_0^{(l)} \\ \vdots \\ \alpha_{N-1}^{(l)} \end{bmatrix} = \frac{1}{2^l}\Psi(u)^t \cdot W^l \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \end{aligned}$$

Now, apply Theorem 2.6 and the proof is complete. □

We now assume $\sum c_{2n} = \sum c_{2n+1} = 1$; then the vector $[1, \ldots, 1]^t$ is a right eigenvector of W with eigenvalue 2. Indeed, for any $0 \le i \le N-1$, the sum of the ith row equals

$$\begin{aligned} \sum_{j=0}^{N-1} W_{i,j} &= \sum_{j=0}^{N-1}\sum_{m=-N}^{N} c_{i+m}c_{m\pm 2j} = \sum_{j=-(N-1)}^{N-1}\sum_{m=-N}^{N} c_{i+m}c_{m+2j} \\ &= \sum_{m=-N}^{N} c_{i+m}\left(\sum_{j=-(N-1)}^{N-1} c_{m+2j}\right) = \sum_{m=-N}^{N} c_{i+m} = 2. \end{aligned}$$

Also, $[1, \ldots, 1]W^t u = [2, \ldots, 2]u = 2\sum u_i$ implies that W^t is invariant on H.

Recall that the *algebraic multiplicity* of an eigenvalue λ_0 is the order of the factor $(\lambda - \lambda_0)$ in the characteristic polynomial. We can now state and prove our main theorem of this section.

THEOREM 4.4. *Suppose $\sum c_{2m} = \sum c_{2m+1} = 1$. If the eigenvalue 2 of W is of algebraic multiplicity 1, and all other eigenvalues of W are less than 2 in absolute value, then equation (2.1) has nonzero L_c^2-solutions.*

Proof. For any $u = [u_0, \ldots, u_{N-1}]^t \in H(\tilde{v}) \subseteq H$,

$$\sum_k \sum_{|i-j|=k} u_i u_j = \left|\sum u_j\right|^2 = 0,$$

so $\Psi(u) \in H$. It follows from the assumption that the eigenvalues of W on H are less than 2 that we have for $w \in H$,

$$\lim_{l \to \infty} \frac{1}{2^l} w^t \cdot W^l = 0.$$

Theorem 4.4 now follows from Proposition 4.3 directly. □

Remark 7. We can write the matrix W as follows: Let $P = [p_0, \ldots, p_{N-1}]$ be an orthonormal matrix with $p_0 = [\frac{1}{\sqrt{N}}, \ldots, \frac{1}{\sqrt{N}}]^t$; it follows that

$$P^* W P = \begin{bmatrix} 2 & * \\ 0 & W_1 \end{bmatrix},$$

where the $(N-1) \times (N-1)$ matrix W_1 is the restriction of W on H. Theorem 4.4 tells us that equation (2.1) has a nonzero L^2_c-solution if W_1 has spectral radius less than 2.

For the converse of the above theorem we need the following lemma.

LEMMA 4.5. *The image of H under the map Ψ contains an $(N-1)$-dimensional region of H.*

Proof. This follows from the observation that the vectors

$$[2, -2, 0, \ldots, 0]^t, \ [2, 0, -2, 0, \ldots, 0]^t, \ \ldots, \ [2, 0, \ldots, 0, -2]^t$$

are the images of

$$[1, -1, 0, \ldots, 0]^t, \ [1, 0, -1, 0, \ldots, 0]^t, \ \ldots, \ [1, 0, \ldots, 0, -1]^t$$

under the continuous map Ψ. □

PROPOSITION 4.6. *Suppose $\sum c_{2n} = \sum c_{2n+1} = 1$ and (2.1) has a nonzero L^2_c-solution f. Let $v = [f_{[0,1]}, \ldots, f_{[0,N-1]}]^t$ be the average vector of f; if $H(\tilde{v}) = H$ and $\{W^k e_1\}_{k=1}^N$ spans $\mathbf{R}^N$, then the eigenvalue 2 of W has algebraic multiplicity 1, and all other eigenvalues are less than 2 in absolute value.*

Proof. If $\{W^k e_1\}$ spans $\mathbf{R}^N$, then (4.2) is equivalent to

$$\lim_{l \longrightarrow \infty} \frac{1}{2^l} \Psi(u)^t \cdot W^l = 0$$

for any $u \in H(\tilde{v})$. But if $H(\tilde{v}) = H$, then by Lemma 4.5, $\Psi(H)$ is also a $(N-1)$-dimensional region contained in H. So the spectral radius of W_1 must be less than 2 and the proposition follows. □

Remark 8. If we impose the m-sum rules (2.7), then for any $u \in H_m$, we also have $\Psi(u) \in H_m$. This is true because for any $j = 0, 1, \ldots, m$,

$$\Psi(u)^t \cdot \begin{bmatrix} 0^j \\ 1^j \\ \vdots \\ (N-1)^j \end{bmatrix} = \sum_{k=0}^{N-1} k^j \left(\sum_i u_i u_{i+k} + \sum_i u_i u_{i-k} \right)$$

$$= \sum_i u_i \left(\sum_{k=0}^{N-1} k^j u_{i+k} \right) + \sum_i u_i \left(\sum_{k=0}^{N-1} k^j u_{i-k} \right),$$

which is 0 since it can be shown inductively that $\sum_{k=0}^{N-1} k^j u_{i\pm k} = 0$ for $j = 0, 1, \ldots, m$ and any i. By Theorem 2.12, Proposition 4.3 reduces to the following corollary.

COROLLARY 4.7. *Suppose that $\sum c_n = 2$ and the m-sum rules hold, and suppose there is a 2-eigenvector v of $(T_0 + T_1)$ such that for any $u \in H_m$,*

$$\lim_{l \longrightarrow \infty} \frac{1}{2^l} u^t \cdot W^l \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0.$$

Then equation (2.1) *has a nonzero L^2_c-solution.*

Remark 9. In [E], [Her1], [Her2], and [V] there are various characterizations of the existence of L^2_c-solutions; the Sobolev exponents and energy moments are also obtained. In those papers the Fourier method was used; the dilation equation (2.1) becomes

$$\hat{f}(\xi) = m_0 \left(\frac{\xi}{2}\right) \hat{f}\left(\frac{\xi}{2}\right),$$

where $m_0(\xi) = \frac{1}{2} \sum c_k e^{-ik\xi}$. Let $g(\xi) = \sum_{k=-\infty}^{\infty} |\hat{f}(\xi + 2\pi k)|^2$; then

$$g(\xi) = \left|m_0\left(\frac{\xi}{2}\right)\right|^2 g\left(\frac{\xi}{2}\right) + \left|m_0\left(\frac{\xi}{2} + \frac{\pi}{2}\right)\right|^2 g\left(\frac{\xi}{2} + \frac{\pi}{2}\right). \tag{4.3}$$

Villemoes [V] showed that a nonzero L^2_c-solution exists if and only if there is a nonnegative trigonometric polynomial $g(\xi) = \sum_{k=-(N-1)}^{N-1} a_k e^{-ik\xi}$ satisfying $g(0) > 0$ and (4.3).

For $g(\xi) = \sum_{k=-(N-1)}^{N-1} a_k e^{-ik\xi}$, it follows from a direct calculation that equation (4.3) is equivalent to the fact that $[a_{-(N-1)}, \ldots, a_{N-1}]$ is a left 2-eigenvector of W', where W' is a $(2N-1) \times (2N-1)$ matrix with (i, j) entry equal to

$$\sum_{-\infty}^{\infty} c_{i+m} c_{m+2j}, \qquad -(N-1) \leq i, j \leq N-1.$$

By the symmetry of W' and the fact that the c_k's are real, one can reduce the operator W' to the matrix W we consider here (see Remark 3.2 in [V]).

Lawton [La] showed that the scaling function f generates an orthonormal basis of L^2 if and only if the vector $[a_{-(N-1)}, \ldots, a_{N-1}]$ with $a_k = \delta_{0,k}$ is the only left eigenvector of W' corresponding to eigenvalue 2.

Hervé [Her1] and [Her2] used an iteration argument based on (4.3) (with 2 replaced by p) to determine the condition for the existence of the solution whose Fourier transform $\hat{f}$ is in L^p. He also calculated the Sobolev exponents

$$s_p = \sup\left\{s \geq 0 : \quad \int |\hat{f}(\xi)|^p (1 + |\xi|^{ps}) d\xi < \infty\right\}$$

for such f.

To conclude this section we will demonstrate the foregoing results for the case $N = 3$. By Lemma 4.1, we calculate that

$$W = \begin{bmatrix} 2(1-\delta) & 2\delta & 0 \\ 1 - c_0 c_3 & 1 & c_0 c_3 \\ \delta & 2(1-\delta) & \delta \end{bmatrix},$$

where $\delta = c_0 - c_0^2 + c_3 - c_3^2$. In addition to 2, W has two eigenvalues

$$\frac{1}{2}\left(1 - c_0 + c_0^2 - c_3 + c_3^2 \pm \sqrt{1 + 26c_0c_3 - (c_0+c_3)\omega_1 + (c_0^2+c_3^2)\omega_2}\right),$$

where

$$\omega_1 = (18c_0^2 + 18c_3^2 - 16c_0c_3 + 6), \quad \omega_2 = (9c_0^2 + 9c_3^2 + 16c_0c_3 + 15).$$

It follows from Theorem 4.4 that if the two eigenvalues are less then 2, then L_c^2-solutions exist.

For $N = 3$, if we adopt the approach in §3 by reducing the matrices T_i on H to S_i on $\mathbf{R}^2$, $i = 0, 1$, then the above analysis is more transparent and the result can be sharpened.

Let $\tilde{M}_0$ be the 2×2 identity matrix. Assume

$$\tilde{M}_k = \sum_{|J|=k} S_J^t S_J = \begin{bmatrix} \alpha^{(k)} & \beta^{(k)} \\ \beta^{(k)} & \alpha^{(k)} \end{bmatrix}.$$

A direct computation shows that

$$\begin{aligned} \tilde{M}_{k+1} &:= \sum_{|J|=k+1} S_J^t S_J = S_0^t \tilde{M}_k S_0 + S_1^t \tilde{M}_k S_1 \\ &= \begin{bmatrix} (c_0^2 + c_3^2 + d^2)\alpha^{(k)} - 2c_0c_3\beta^{(k)} & -(c_0d + c_3d)\alpha^{(k)} + (c_0d + c_3d)\beta^{(k)} \\ -(c_0d + c_3d)\alpha^{(k)} + (c_0d + c_3d)\beta^{(k)} & (c_0^2 + c_3^2 + d^2)\alpha^{(k)} - 2c_0c_3\beta^{(k)} \end{bmatrix}, \end{aligned}$$

where $d = (1 - c_0 - c_3)$. Comparing the first columns of the two matrices $\tilde{M}_k$ and $\tilde{M}_{k+1}$, we can define the matrix $\tilde{W}$ as follows:

$$\begin{bmatrix} \alpha^{(k+1)} \\ \beta^{(k+1)} \end{bmatrix} = \tilde{W} \begin{bmatrix} \alpha^{(k)} \\ \beta^{(k)} \end{bmatrix} = \tilde{W}^{k+1} \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

where

$$\tilde{W} = \begin{bmatrix} c_0^2 + c_3^2 + d^2 & -2c_0c_3 \\ -d(c_0+c_3) & d(c_0+c_3) \end{bmatrix}.$$

A direct computation shows that $\tilde{W}$ has the same eigenvalues as W_1 in Remark 7; however, we do not known their exact relationship.

THEOREM 4.8. *Suppose $c_0 + c_2 = c_1 + c_3 = 1$; then the dilation equation* (2.1) *with $N = 3$ has nonzero L_c^2-solutions if and only if either* $(c_0, c_3) = (1, 1)$ *or the matrix*

$$\begin{bmatrix} c_0^2 + c_3^2 + d^2 & -2c_0c_3 \\ -d(c_0+c_3) & d(c_0+c_3) \end{bmatrix},$$

where $d = (1 - c_0 - c_3)$, has spectral radius less than 2.

Proof. For any $u = [x, y]^t$, we have

$$\begin{aligned} \frac{1}{2^l}\sum_{|J|=l} ||S_J u||^2 &= \frac{1}{2^l}\sum_{|J|=l} u^t S_J^t S_J u = \frac{1}{2^l} u^t \left(\sum_{|J|=l} S_J^t S_J\right) u \\ &= \frac{1}{2^l} u^t \tilde{M}_l u = \frac{1}{2^l}[x, y] \begin{bmatrix} \alpha^{(l)} & \beta^{(l)} \\ \beta^{(l)} & \alpha^{(l)} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \frac{1}{2^l}[x^2 + y^2, 2xy] \begin{bmatrix} \alpha^{(l)} \\ \beta^{(l)} \end{bmatrix} \\ &= \frac{1}{2^l}[x^2 + y^2, 2xy]\tilde{W}^l \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{aligned}$$

Since $\{[x^2+y^2, 2xy]; \quad [x,y] \in \mathbf{R}^2\}$ spans $\mathbf{R}^2$, the condition

$$\frac{1}{2^l}\sum_{|J|=l} ||S_J u||^2 \longrightarrow 0 \quad \text{as} \quad l \longrightarrow \infty \quad \text{for all} \quad u \in \mathbf{R}^2$$

is equivalent to

$$\frac{1}{2^l}\tilde{W}^l \begin{bmatrix} 1 \\ 0 \end{bmatrix} \longrightarrow 0 \quad \text{as} \quad l \longrightarrow \infty. \tag{4.4}$$

We will show that (4.4) holds if and only if the spectral radius of $\tilde{W}$ is less than 2; hence Theorem 3.6 applies and we are done.

If $d(c_0+c_3) \neq 0$, let $u_1 = \tilde{W}[1,0]^t$; then u_1 and $[1,0]^t$ are linearly independent. (4.4) is equivalent to the statement that $\frac{1}{2^l}\tilde{W}^l u \longrightarrow 0$ for all $u \in \mathbf{R}^2$, and hence $\frac{1}{2^l}\tilde{W}^l \longrightarrow 0$ as $l \longrightarrow \infty$. This means that the eigenvalues of W are less than 2.

If $d(c_0+c_3) = 0$, then $\tilde{W}$ is of the form

$$\begin{bmatrix} w_1 & w_2 \\ 0 & 0 \end{bmatrix}$$

for some w_1 and w_2. It follows that

$$\tilde{W}^l = \begin{bmatrix} w_1^l & w_1^{l-1}w_2 \\ 0 & 0 \end{bmatrix},$$

and (4.4) implies $|(1/2^l)w_1^l| < 1$, that is, $|w_1| < 2$. Again the eigenvalues of W are all less than 2 in absolute value. □

Appendix. For the four-coefficient dilation equation

$$f(x) = c_0 f(2x) + c_1 f(2x-1) + c_2 f(2x-2) + c_3 f(2x-3) \tag{A1}$$

with $c_0 + c_2 = 1$, $c_1 + c_3 = 1$, let c_0, c_3 be the independent parameters. We use the **Mathematica** on a NeXT workstation to plot the following regions of (c_0, c_3), for which the compactly supported L^1 and L^2 solutions exist.

Let D_l be the regions of (c_0, c_3) for which

$$\frac{1}{2^l}\sum_{|J|=l} ||S_J u|| < 1 \quad \text{for} \quad u \in \mathbf{R}^2, \ ||u|| \leq 1 \tag{A2}$$

holds. By Theorem 3.6, except for $(c_0, c_3) = (1,1)$, equation (A1) has a nonzero compactly supported L^1-solution if and only if (c_0, c_3) is in the union of the regions D_l, $l = 1, 2, \ldots$. In Fig. 1 we display the regions D_l for $l = 1, 2, 4$, and 8. Here the norm is $||[x,y]^t|| = |x| + |y|$.

Note that the regions are increasing (Remark 5, Theorem 2.6). When $l = 1$ and 2, condition (A2) can be written as

$$|c_0| + |c_3| + |1 - c_0 - c_3| < 2,$$

and

$$c_0^2 + c_3^2 + (1-c_0-c_3)^2 + |c_0(1-c_0)| + |c_3(1-c_3)| + |c_0(1-c_0-c_3)| \\ + |c_3(1-c_0-c_3)| < 4,$$

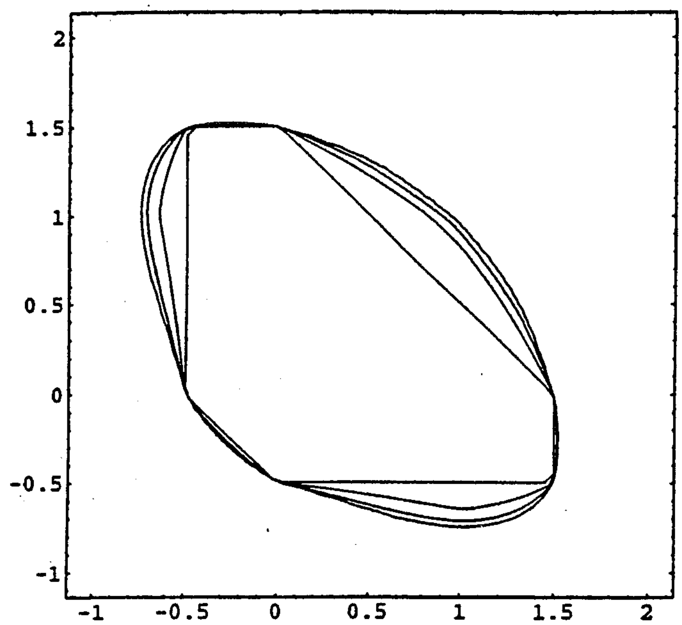

FIG. 1.

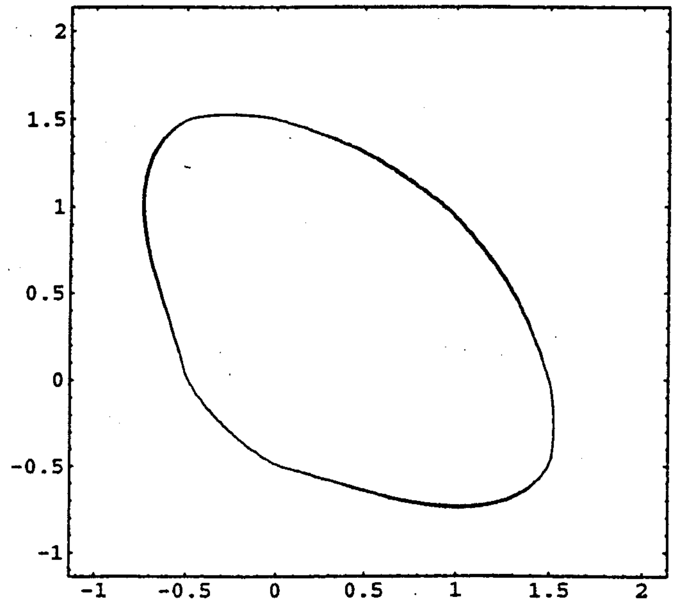

FIG. 2.

respectively. For $l \geq 3$, the expression is more tedious.

In Fig. 2 we plot the regions D_l for $l = 6$ and 8. Note that they are very close, and hence they are good approximations of the admissible region of (c_0, c_3) for L^1-solutions.

In Fig. 3 we plot the following regions of (c_0, c_3) for the existence of the L^1-solutions from some previous results.

The region outside the ellipse

$$c_0^2 + c_3^2 - c_0 - c_3 + c_0c_3 = 1$$

is known to have no L^1-solution for (A1).

The region bounded by the dotted line

$$c_0^2 + c_3^2 + |c_0(1 - c_0)| + |c_3(1 - c_3)| + 2|1 - c_0 - c_3| < 4$$

is a sufficient condition given by Pan [P].

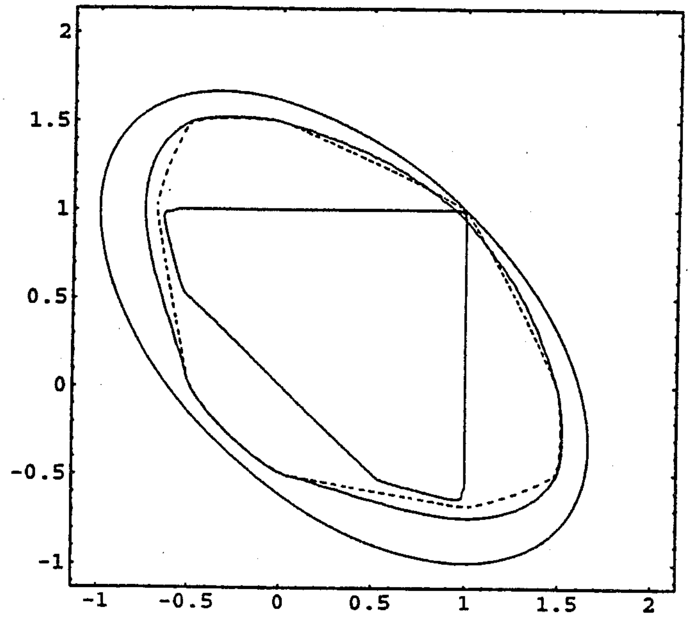

FIG. 3.

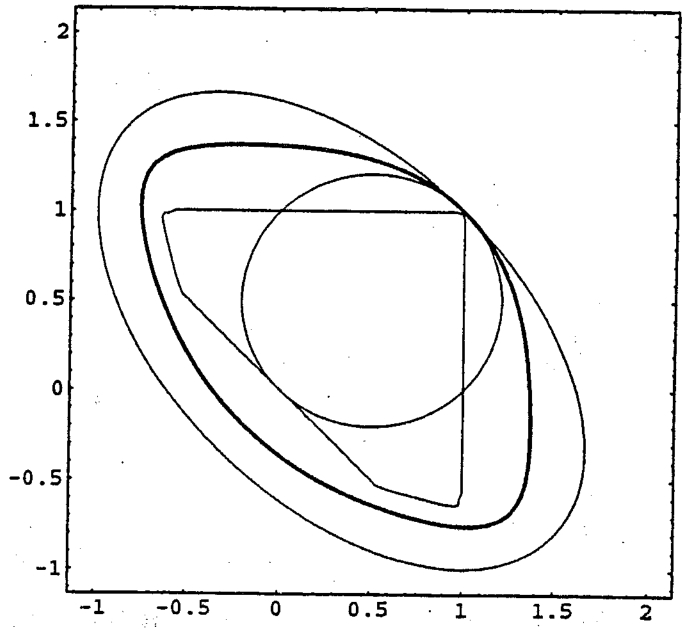

FIG. 4.

The region D_8 is determined by (A2) with $l = 8$.

Also, the triangular-shaped region approximates the domain where the joint spectral radius of T_0 and T_1 is less than 1, hence nonzero compactly supported continuous solutions exist there.

In Fig. 4, we plot the following regions:

First we plot the region determined by the ellipse as in Fig. 3.

Next we plot the region bounded by the thicker line consisting of points (c_0, c_3) for which the matrix

$$\begin{bmatrix} c_0^2 + c_3^2 + d^2 & -2c_0c_3 \\ -d(c_0+c_3) & d(c_0+c_3) \end{bmatrix}, \quad \text{where} \quad d = 1 - c_0 - c_3,$$

has spectral radius less than 2. This is a necessary and sufficient condition for (A1) to have nonzero compactly supported L^2-solutions with one exception: $(c_0, c_3) = (1, 1)$ (Theorem 4.8).

We also plot the region for the existence of compactly supported continuous solutions as in Fig. 3.

Finally, we plot the circular region

$$(c_0 - 1/2)^2 + (c_3 - 1/2)^2 \leq 1/2,$$

a sufficient condition of the existence of L^2-solutions given in [La]. The boundary is called the *circle of orthogonality*: if the wavelet generated by the scaling function satisfying (A1) is orthonormal, then the point (c_0, c_3) must be on the circle.

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